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AMG #24

AMP Memo 20.7

MATHEMATICAL THEORY OF THE INFLUENCE OF A DOME ON THE DIRECTIVITY PATTERN OF SOUND BEAMS

PART IV

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A Memorandum Submitted By The
Applied Mathematics Groups
of
Columbia University & New York University
to the
Applied Mathematics Panel
National Defense Research Committee

September 1943

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Mathematical Study of Effects on Sound Transmission Arising from the Use of Oil in Domes

1) Introduction. In previous studies* we have considered various effects on the propagation of sound which arise when the sound source is enclosed by a thin shell or dome made of material differing from water in acoustic properties. The dome was, however, always considered to be filled with water. In the present memorandum we propose to study the effects which arise when the dome is filled with a liquid (usually oil) differing from water in acoustic properties.

We begin the present study by considering the following special problems: 1) The case of a plane wave incident on a plane metal sheet, with oil (or some other liquid) on one side of the sheet and water on the other, 2) The case of a spherical wave and the combination of a spherical shell of metal with oil inside and water outside.

The main conclusion which can be drawn from these special cases is that there are oils which could be used without causing appreciable disturbances. In case 2 for spheres of not too small radius, and in case 1 when the plane

* Mathematical Theory of the Influence of a Dome on the Directivity Pattern of Sound Beams - AMP Memo 20.1 and 20.4

wave is incident normally to the plane separating the oil and water, it is sufficient to match the values of the acoustic impedances ρc of oil and water in order to avoid disturbances from use of an oil. However, for non-normal incidence in case 1, it is necessary to match the sound velocities as well as the impedances, or, what comes to the same thing, to match both ρ and c values in order to avoid disturbances for all angles of incidence. The formulas also indicate that if it is not possible to match both ρ and c values, then it is best to choose an oil for which the sound velocity c has a value slightly higher than that of water, but with a density slightly lower than that of water. Oils with such properties exist; numerical calculations indicate that very slight disturbances would result from their use even in case 1 at angles of incidence up to and even greater than 80°.* Two such liquids are Liquid E. C. and Lubric Oil X200 at 25°C; the values of the acoustic constants for these liquids were taken from a memorandum prepared by J. B. Johnson and W. P. Mason of the Bell Telephone Laboratories. At temperatures lower than 25°C the matching is not as close; nevertheless these oils would probably cause no great disturbances even at 0°C, particularly since the sound velocity tends to increase relative to water as the temperature falls.

* However, if the oil should contain air bubbles in any quantity, large disturbances could be expected.

In the last sections a general theory, using the perturbation method, is developed for a dome of any shape and a beam of any given character. This theory is valid only for oils with acoustic properties differing not too much from those of water. The results of the approximate theory are checked with those of the exact theory for the case of a spherical wave and a concentric sphere of oil immersed in water. The approximate theory is then applied to the case of a spherical dome filled with oil and a sharply defined directional beam. Unless the c's are very closely matched the disturbance may be quite appreciable in this case.*

2) Plane Wave Incident On a Plane Sheet Separating Oil and Water. The sheet and the liquid media of different properties are indicated in Fig. 1

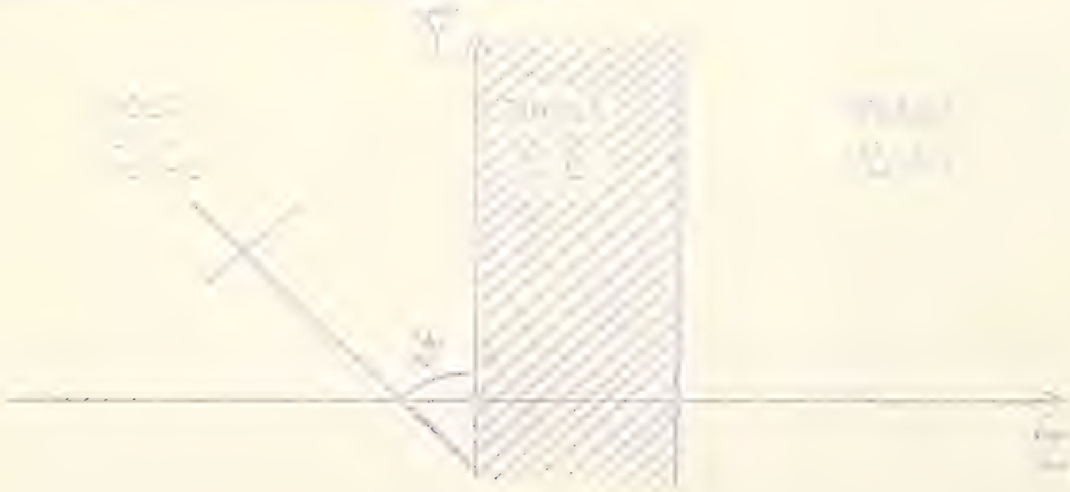

Fig. 1

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* The variation of c with temperature is likely to be important. Thus Lubric oil, for which the disturbance is small at 25°C, yields a large disturbance near 0°C.

Throughout this memorandum we shall denote by the subscripts o, 1, and 2 quantities referring respectively to water, the material of the sheet or shell, and oil.

As is well known, the solution for the acoustic pressure amplitude for a plane wave advancing from the left can be written in this case:

$$(2.1) \qquad p = \begin{cases} e^{i(\beta y + \gamma_2 z)} + b_2 e^{i(\beta y - \gamma_2 z)}, & z < 0 \\ a_1 e^{i(\beta y + \gamma_1 z)} + b_1 e^{i(\beta y - \gamma_1 z)}, & 0 < z < h \\ a_0 e^{i(\beta y + \gamma_0 z)}, & h < z . \end{cases}$$

The constants β, γ_o, γ_1, γ_2 are subject to the following relations

[illegible]

in which the c_ν are the sound velocities. The constants a_ν and b_ν are to be determined through the transition conditions.*

---

* The symbol $\left[f \right]_{0-}^{0+}$ means that the difference $f(0+) - f(0-)$ is to be taken.

$$(2.3) \qquad \Big[p\Big]_-^+ = \left[\frac{1}{\rho}\frac{\partial p}{\partial z}\right]_-^+ = 0 \text{ at } z = 0 \text{ and } z = h \quad ,$$

which state that the pressure and the normal component of the acceleration are continuous at $z = 0$ and $z = h$.

We are mainly interested in comparing the amplitude $|a_0|$ of the transmitted wave with that of the original oncoming wave, which we assumed (as one observes [illegible]) have the amplitude unity. From (2.5) we find that

$$a_0 = \frac{2e^{-i\gamma_0 h}}{(1+\frac{\gamma_0\rho_2}{\gamma_2\rho_0})\cos\gamma_1 h - i(\frac{\gamma_0\rho_1}{\gamma_1\rho_0} + \frac{\gamma_1\rho_2}{\gamma_2\rho_1})\sin\gamma_1 h}$$

from which we have

$$|a_0| = \frac{2}{\left|(1+\frac{\gamma_0\rho_2}{\gamma_2\rho_0})\cos\gamma_1 h - i(\frac{\gamma_0\rho_1}{\gamma_1\rho_0} + \frac{\gamma_1\rho_2}{\gamma_2\rho_1})\sin\gamma_1 h\right|}$$

The departure of $|a_0|$ from unity is then a measure of the disturbance created by the oil and the metal sheet together.

In the following table we give some numerical values for $|a_0|$ when sheets of aluminum and steel, 0.1 cm. thick, are used in combination with various liquids and at a number of angles of incidence. The frequency was assumed to be 24,000 cycles/second. The table also indicates the angle at which total reflection (T.R.) begins. Normal incidence is given by 0° angle of incidence. The results given here are valid for a temperature of 25°C.

[illegible]		[illegible]			
Oil	Metal	[illegible]	[illegible]	70°	T.R. begins at
[illegible]	Steel	[illegible]	[illegible]	[illegible]	60°
	Aluminum	[illegible]	[illegible]	[illegible]	
Water	Steel	[illegible]	[illegible]	[illegible]	No T.R.
	Aluminum	[illegible]	[illegible]		
Diethyl Phthalate	Steel	.94	1.07	T.R.	65°
	Aluminum	1.00	1.10	T.R.	
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	60°
	[illegible]	[illegible]	[illegible]	[illegible]	
Fluolube	[illegible]	.955	T.R.	T.R.	80°
	Aluminum	.990	T.R.	T.R.	
Liquid E.C.	Steel	.90	.905	.625	No T.R.
	Aluminum	.95	.932	.625	
Lubric Oil X200	Steel	.94	.998	1.05	Nearly 90°
	Aluminum	1.00	1.04	1.03	

The main conclusion to be drawn from the table has already been stated in the introduction. The table indicates that there are at least two oils, i.e., Lubric Oil X200 and Liquid E.C., which cause only slight disturbances in the special case under discussion here. One of these oils, Lubric Oil X200, causes little or no disturbance

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even at nearly glancing angles. It should be remembered, however (see section 7 of Part III), that the metal sheet will cause considerable disturbances for angles of incidence greater than 80°, though the metal sheet alone never causes total reflection.

8. The Spherical Dome Containing Oil Inside. Effect of the Combination on a Spherical Wave. In this section we consider the effect of a spherical dome filled with oil on a concentric spherical wave (Fig. 2).

Fig. 2

The solution for the pressure amplitude P can be written in the form

$$(8.1)\quad r\,P = \begin{cases} a_0 e^{ik_0 r} + b_0 e^{-ik_0 r} & , \; r < R_1 \\ a_1 e^{ik_1 r} + b_1 e^{-ik_1 r} & , \; R_1 < r < R_2 \\ a_2 e^{ik_0 r} & , \; R_2 < r \; . \end{cases}$$

The constants a_1 and b_1 are determined from the transition condition:

$$(3.2) \qquad \left[rP \right] = \left[\frac{1}{\rho} \frac{dP}{dr} \right] = 0 \text{ at } r = R_1 \text{ and } r = R_2 \, .$$

We wish to compare the resulting amplitude in the region outside the sphere (i.e. in water) with the amplitude when the dome is absent and the entire space is filled with water, that is, with

(3.3) [illegible]

As in previous cases, we assume that the strength of the singularity at $r = 0$ is maintained when the dome is present; consequently the amplitudes given by (3.3) are to be compared with those given by (3.1) only after setting $a_2 + b_2$ equal to unity. The same effect can be achieved, of course, by dividing the solution given in (3.1) by $a_2 + b_2$. The quantity of principal interest to us is then $\left| \frac{a_0}{a_2 + b_2} \right|$; the magnitude of the disturbance in amplitude is then $\left| \frac{a_0}{a_2 + b_2} \right| - 1$.

From (3.1) and (3.2) we find

$$(3.4) \qquad \frac{a_0}{a_2+b_2} = \frac{e^{-ik_0R}, \; e^{-ik_0h}}{A \cos k_1 h + B \sin k_1 h} \, ,$$

in which

$$[illegible]$$

$$[illegible] \sin k_2 R_1 + [illegible] \cos k_2 R_1 ,$$

with

$$[illegible]$$

We now make the assumption that the thickness h [illegible] R_1 and R_2 are rather large [illegible] $\sin k_1 h \approx k_1 h$, [illegible] can be neglected. With these assumptions (5.4) takes the simpler form:

$$(5.5) \quad \left| \frac{\Delta p}{p} \right| = \frac{1 - \frac{1}{2} k_0 h \left\{ [illegible] \, 1 + (\ell_2^2 - 1) \sin^2 k_2 R_1 \right\} \sin 2 k_2 R_1}{\sqrt{1 + (\ell_2^2 - 1) \sin^2 k_2 R_1}}$$

in which

$$(5.6) \quad v = [illegible] = [illegible] \quad \text{and}$$

$$(5.7) \quad \ell_2 = \frac{\rho_2 k_0}{\rho_0 k_2} = \frac{\rho_2 c_2}{\rho_0 c_0}$$

We observe that when $\ell_2 = 1$, i.e. when the acoustic impedance of the oil inside the dome is the same as that for the water outside the relative disturbance in the pressure amplitude

has the same maximum value as in the case when the dome is filled with water. In other words, the disturbances due to filling a dome with oil can be minimized by matching impedances alone. For the shell of the dome, as we have seen, it is necessary to match the values of density and sound velocity.

As one would expect, the amplitude disturbances created by the use of oil are quite small if the impedances are closely matched. This is indicated in the following table, which compares the values of $\left| \frac{a_0}{a_2 + b_2 i} \right|$ when Fluolube is used inside the dome instead of water and for both steel and aluminum domes 1.2 cm. in thickness. The values given hold at a temperature of 25°C. It is to be noted that the acoustic impedance of Fluolube differs from that of sea water by about one per cent, but that its density is twice that of water. In spite of this, the disturbances are nearly the same whether Fluolube is used inside the dome in place of water, or not.

For the case of a spherical wave and spherical shell, we conclude that an oil of impedance near to that of water may be used inside the shell without creating an appreciable disturbance.

It has some interest to specialize the formula (3.5) for the case in which the metal shell is not present, in order to obtain that part of the disturbance which is due solely to the liquid inside the shell. For this purpose we need only set $h = 0$ and $R_1 = R_2 = R$. The result is (after a slight rearrangement of the denominator and use of $e_1 = (\frac{k}{k_1})$ -

$$\left| \frac{a_0}{a_2 + b_2 i} \right| = \frac{2}{\sqrt{2 + 2\left(\frac{\rho_2 c_2}{\rho_0 c_0}\right)^2 + 2\left[1 - \left(\frac{\rho_2 c_2}{\rho_0 c_0}\right)^2\right] \cdot \cos 2k_2 R}}$$

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Fluid Inside	[illegible]	Aluminum
Water	$1-.373 \sin 2 k_3 R_1$	$1-.150 \sin 2 k_3 R_1$
[illegible]	$\dfrac{1-.373 \dfrac{.997}{[illegible]\ 2 k_2 R_1} \sin 2 k_2 R_1}{\sqrt{1-.007 \sin^2 k_2 R_1}}$ $\approx 1-.37 \sin 2 k_2 R_1$	$\dfrac{[illegible]}{\sqrt{1-.007 \sin^2 k_2 R_1}}$ $\approx 1-.15 \sin 2 k_2 R_1$

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We observe that the value of the right hand side is unity, if the ρc values for oil and water are matched, that is, the disturbance in amplitude would be zero.* Otherwise the disturbance in amplitude varies in a rather sensitive way when $k_o R$ is changed. The extreme values for $\frac{a_o}{\cdots}$ are readily seen to be those for which $\cos 2k_o R$ is $+1$ or -1 and these values are

$$(5.9) \qquad \left| \frac{a_o}{r_o b_o} \right| = \left\{ \begin{array}{l} 1 \\ \frac{\rho_o c_o}{\rho_w c_w} \end{array} \right. \quad \text{[illegible]}$$

Thus the extreme values for the amplitude disturbance are given by $\left| \frac{\rho_o c_o}{\rho_w c_w} - 1 \right|$. In other words, the disturbance can be kept low by matching impedances alone--unlike the case of the disturbances due to the shell of the dome in which it is necessary to match both density and sound velocity in order to reduce the disturbance to zero.

We append a table showing the impedance ratios of various liquids, and the extreme values for the percentage amplitude disturbance (at a temperature of 25°C):

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*This result is not surprising [illegible] that the same is true for the case of oil and water separated by a plane (with normal incidence). We have already made use of the assumption that R is large and this, together with the fact that we deal with a concentric spherical wave makes the comparison obvious.

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Material	$\frac{\rho_0 c_0}{\rho_2 c_2}$	% Amplitude Disturbance (Extreme Values)
Kerosene	1.45	[illegible]
Castor Oil	1.036	[illegible]
Flualube	0.992	[illegible]
Diethyl Phthalate	0.984	[illegible]
Mercury	0.0835	[illegible]

Apparently it is possible to select an oil to fill the dome which will cause only slight disturbance, at least for the case of a spherical wave and spherical dome.

4) Perturbation Method for the Oil-filled Dome of Any Shape. It will be assumed that the liquid used to fill the dome differs only slightly from water in its acoustic properties. We take advantage of this assumption to set up a perturbation procedure, which consists in a power series development with respect to the difference $\bar{K} = k_2 - k_0$ in the values of the constants, $k_0 = \frac{\omega}{c_0}$ and $k_2 = \frac{\omega}{c_2}$ in water and oil respectively. This is exactly analogous to the procedure used for treating disturbances due to the dome; in the latter case the solutions were developed with respect to the thickness h of the dome. One could set up a perturbation procedure in which disturbances due to the dome as well as to use of oil inside the dome would be treated simultaneously; that is, a development with respect to both $\bar{K}$, and h, the thickness of the dome shell, could be obtained. If, however, only the linear terms in the development are desired,

they can be obtained through addition of the linear terms in the developments in series with respect to K and h separately. In other words, the effects due to the use of oil inside the dome can be considered alone, the dome being considered absent. Consequently, in this section we proceed as though the dome itself were not present.

We make the same general fundamental assumptions here as were made for the treatment of the dome problem, except those referring specifically to the shell of the dome. The quantity of fundamental interest is the complex pressure amplitude $P(x,y,z)$, which satisfies the differential equation (equation (4.3) of AMP Memorandum No. 20.1),

$$\nabla^2 P + k^2 P = 0 \quad , \quad k^2 = \omega^2/c^2 \tag{4.1}$$

in any region in which k is constant. As in the earlier work, we assume always that P behaves like $C \cdot \frac{e^{ikr}}{r}$ at ∞ corresponding to the physical assumption of an outgoing wave at ∞.

We assume that the whole of space is filled with water except for a bounded region R of surface S (Fig. 3) which is filled with a different medium of constant acoustic properties.

Fig. 3

In water the constant k and the density ρ are denoted by k_0 and ρ_0 ; inside R these quantities are denoted by k_2 and ρ_2 (to distinguish them from the corresponding quantities k_1 and ρ_1 which were always used for the material of the dome).

We are interested in determining the pressure P when sound sources of a definite character are prescribed inside R. The boundary value problem to be solved can then be formulated as follows: The solution P of (4.1) with $k = k_2$ inside R and $k = k_0$ outside R which has prescribed singularities inside R and the proper behavior at ∞ is wanted. In addition, P should be continuous on crossing S, as well as the quantity $\frac{1}{\rho}\frac{\partial P}{\partial n}$, $\frac{\partial P}{\partial n}$ meaning the derivative of P with respect to distance along the normal to S. These are, of course, the same transition conditions as were used in the previously quoted Memorandum; in the jump notation, they are written

$$(4.2) \qquad \left[P\right]_-^+ = 0 \;, \quad \left[\frac{1}{\rho}\frac{\partial P}{\partial n}\right]_-^+ = 0 \;.$$

In order to carry out the perturbation scheme we set

$$(4.3) \qquad P = p + \kappa q_2 \;,$$

in which p is assumed to be the prescribed "original" wave when the entire space is considered to be homogeneous and

filled with water, and $\kappa\, q_2$ is the disturbance resulting from the introduction of oil instead of water into the region R. Consequently, the function p satisfies everywhere the differential equation

$$\nabla^2 p + k_0^2\, p = 0 \tag{4.4}$$

except at the sound sources, where appropriate singularities are prescribed. The quantity κ is assumed given by

$$\kappa = k_2 - k_0 \; ; \tag{4.5}$$

in other words κ is the difference in the values of k in the two media, as stated earlier.

Upon introducing (4.3) into (4.1) and making use of (4.4), it is readily seen that q_2 satisfies the differential equation

$$\begin{aligned} \nabla^2 q_2 + k_0^2\, q_2 &= \frac{(k_0^2 - k_2^2)\, p}{\kappa} && \text{inside } R \; , \\ \nabla^2 q_2 + k_0^2\, q_2 &= 0 && \text{outside } R \; . \end{aligned} \tag{4.6}$$

We assume q_2 to be regular everywhere except on S. The function q_2 clearly satisfies the following transition conditions obtained from (4.2):

$$(4.7) \qquad \left[q_2 \right]_-^+ = 0$$

$$(4.8) \qquad \left[\frac{1}{\rho} \frac{\partial q_2}{\partial n} \right]_-^+ = -\frac{1}{\rho} \left[\rho \right]_-^+ \frac{\partial p}{\partial n}$$

In obtaining (4.7) and (4.8) no use has been made of the assumed [illegible] on S. Once the original beam p has been prescribed, q_2 is uniquely determined by (4.6) and the transition conditions (4.7) and (4.8) together with the condition that q_2 should behave like an outgoing wave at ∞.

The boundary value problem thus formulated is difficult to solve except in special cases; but it can be replaced by a simpler one obtained by assuming the quantity κ defined in (4.5) to be small, which means in turn that the two media differ only slightly in the values for the sound velocities. It is also necessary to assume that the difference in the densities is of the same order as the difference in the values of k. If we set $k_2 = k_0 + \kappa$ in (4.6) and retain only the terms of zero order in κ, we obtain

$$(4.9) \qquad \begin{aligned} \nabla^2 q_2 + k_0^2 q_2 &= -2k_0 p \qquad &\text{inside } R, \\ \nabla^2 q_2 + k_0^2 q_2 &= 0 \qquad &\text{outside } R, \end{aligned}$$

as differential equation for q_2 which represents now the disturbance only within first order terms in κ. Proceeding in

the same manner with the transition conditions (4.7) and (4.8) we obtain, on retaining only the terms of lowest order in κ :

$$(4.10) \qquad \Big[q_2 \Big]_-^+ = 0 \quad , \text{ and}$$

$$(4.11) \qquad \left[\frac{\partial q_2}{\partial n} \right]_-^+ = \lambda \rho_0 \frac{\partial p}{\partial n} \bigg|_s \quad ,$$

in which λ is defined by

$$(4.12) \qquad \lambda \kappa = \frac{1}{\rho_2} - \frac{1}{\rho_0}$$

The quantity λ is thus proportional to the difference in the reciprocals of the densities of the two media. The condition (4.11) is obtained from (4.8) as follows: Making use of (4.12) in (4.8) we have

$$\frac{1}{\rho_0} \frac{\partial q_2}{\partial n} \bigg|^+ - \frac{1}{\rho_2} \frac{\partial q_2}{\partial n} \bigg|^- = - \frac{1}{\kappa} (-\lambda \kappa) \frac{\partial p}{\partial n} \bigg|_s \quad ,$$

$$\text{or} \quad \frac{1}{\rho_0} \frac{\partial q_2}{\partial n} \bigg|^+ - (\lambda \kappa + \frac{1}{\rho_0}) \frac{\partial q_2}{\partial n} \bigg|^- = \lambda \frac{\partial p}{\partial n} \bigg|_s \quad .$$

Condition (4.11) follows from the last equation upon rejecting the terms in κ and multiplying the equation by ρ_0 .

The unique solution to the boundary value problem formulated in equations (4.9) to (4.12) (with, in addition, the condition at ∞) is given by

(4.13) $$q_2 = q_2^{(1)} + q_2^{(2)} \quad , \text{ with}$$

(4.14) $$q_2^{(1)} = \frac{k_o}{8\pi} \iiint_R \frac{e^{ik_o r'}}{r'}\, p\, dv \quad , \text{ and}$$

(4.15) $$q_2^{(2)} = -\frac{\lambda_o}{4\pi} \iint_S \frac{e^{ik_o r'}}{r'} \frac{\partial p}{\partial n}\bigg|_2 dS$$

that is, by the sum of a volume integral over R and of a surface integral over S . That (4.13) is the solution satisfying all of the conditions can be proved by direct verification, or inferred from the solution of the exactly analogous problem in potential theory.

We see, as one would expect, that the disturbance is zero if $k_2 = k_o$ and $\rho_2 = \rho_o$, i.e. if an oil is used which has the same values for ρ and k (or ρ and c) as water. Otherwise it is clear that the disturbance will be minimized by using oils with acoustic properties as near to those of water as possible.

5) Spherical Wave and Spherical Drop. Comparison of Exact and Approximate Solutions. In this section we compare our approximate solution obtained by the perturbation method developed above with the exact solution for the case of a

spherical wave and concentric spherical dome of radius R filled with oil. We recall that the presence of the dome itself is ignored.

Consider first the approximate solution. Instead of making use of the solution given by (4.13) we prefer to integrate the differential equations (4.9) explicitly. The function p is assumed to be given by

$$p = \frac{e^{ik_0 r}}{r} \tag{5.1}$$

Since the problem possesses radial symmetry, the differential equations (4.9) become ordinary differential equations; they may be written as follows:

$$\frac{d^2}{dr^2}(rq_2) + k_0^2 (rq_2) = \begin{cases} -2k_0 e^{ik_0 r} & , \; r < R \\ 0 & , \; r > R \end{cases} \tag{5.2}$$

The conditions at the boundary surface $r = R$, as obtained from (4.10) and (4.11), are:

$$\Big[rq_2 \Big]_{R_-}^{R_+} = 0 \quad , \quad \text{and} \tag{5.3}$$

$$\frac{1}{R} \left[\frac{d(rq_2)}{dr} \right]_{R_-}^{R_+} = \lambda \rho_0 \left. \frac{dp}{dr} \right|_{R} \quad . \tag{5.4}$$

The general solution of (5.2) is

$$
(5.5) \qquad rq_2 = \begin{cases} a_0 \cos k_0 r + b_0 \sin k_0 r + ire^{ik_0 r} & , \; r < R \\ a_\infty \cos k_0 r + b_\infty \sin k_0 r & , \; r > R \, . \end{cases}
$$

Since q_2 is assumed to be regular everywhere except at $r = R$, we must set $a_0 = 0$ in order to avoid a singularity at $r = 0$. The condition at ∞ (that q_2 should behave like an outgoing wave) is satisfied by taking $b_\infty = ia_\infty$. We have, then

$$
(5.6) \qquad rq_2 = \begin{cases} b_0 \sin k_0 r + ire^{ik_0 r} & , \; r < R \\ a_\infty e^{ik_0 r} & , \; r > R \, . \end{cases}
$$

The constants b_0 and a_∞ are then determined from the conditions (5.3) and (5.4). In particular, the value of a_∞, which clearly yields the disturbance outside the region filled with oil, is found to have the value

$$
(5.7) \qquad a_\infty = i R + i e^{ik_0 R}\left(\frac{\lambda \rho_0}{ik_0 R} - \frac{1}{k_0} - \lambda \rho_0\right) \sin k_0 R \, .
$$

We proceed to find the exact solution of the wave equation (4.1) in this case for the total complex pressure amplitude P. We write the solution in the form

$$
(5.8) \qquad rP = \begin{cases} A_0 e^{ik_2 r} + B_0 e^{-ik_2 r} & , \; r < R \\ A_\infty e^{ik_0 r} & , \; r > R \end{cases}
$$

At $r = R$, the conditions (4.2) are to be satisfied; they are in this case

$$(5.9) \qquad \Big[F \Big]_{R-}^{R+} = 0 , \qquad \Big[\frac{1}{\rho} \frac{dF}{dr} \Big]_{R-}^{R+} = 0$$

We set $A_0 + B_0$ equal to one in order that the singularity at $r = 0$ be the same as for the approximate solution. From (5.9) we may determine any two of the integration constants in (5.8) in terms of the third.

Our purpose in this section is to verify the correctness of the approximate solution for this special case. That is, we compare the exact solution F with the approximate solution $p + \kappa\, q_e$ as given above. We find for the exact solution,

$$(5.10) \qquad A'_{\infty} = \frac{A_\infty}{A_0 + B_0} = \text{[illegible]}$$

in which the constant γ is defined by

$$(5.11) \qquad \gamma = \frac{k_0}{\rho_0} + \frac{\lambda \kappa}{1R} ,$$

λ and κ being the constants defined in (4.5) and (4.12). If (5,10) is then developed in powers of κ the result is found to be

$$(5,12) \qquad A'_{\infty} = \frac{A_\infty}{A_0 + B_0} = 1 + \kappa\, a_{\infty} + \cdots ,$$

in which a_∞ is the quantity given by (5.7). But this means that the approximate solution coincides with the exact solution within first power terms in K, at least for $r > R$. Inside the sphere the same is true, as can be shown by the same process.

In the following table we compare the values for the maximum relative amplitude disturbances as given by the approximate and the exact solutions. (The exact solution was given in section 5 above.)

Fluid (at 25°C)	Exact	Limit
Kerosene	0 to +.45	$.207 \cos(2R+\varepsilon) + .207$
Castor Oil	0 to .086	$.0414 \cos(2R+\varepsilon) + .0414$
Flulube	0 to -.008	$.21 \cos(2R+\varepsilon) + .21$
Diethyl Phthalate	0 to .016	$.0065 \cos(2R+\varepsilon) + .0065$
DB + DMP	0 to -.025	$.0097 \cos(2R+\varepsilon) - .0097$

We see that the limit solution yields quite accurate results except in the case of Flulube which has a density twice that of water. Our perturbation procedure could not be expected to yield good results in that case.

6) Spherical dome with a directional beam. In this section we apply the theory developed in section 4 to obtain an estimate for the disturbance due to using oil inside a spherical dome when the original beam p is given in spherical

coordinates by an expression of the form

$$(6.1) \qquad p = \frac{e^{ikr}}{r} \, f(\varphi, \vartheta) .$$

This expression represents only a rough approximation to the actual cases of directional beams, since it would be valid in general only at considerable distances from the sound sources.

The total acoustic pressure amplitude $P = p + K \, q_2$ (see equation 4.3) is known (within first order terms in K) [illegible] q_2 has been determined. We recall that

$$[illegible] ;$$

in which c_2 and c_0 are the sound velocities in oil and water, respectively. The solution for q_2 has been given in equations (4.13), (4.14) and (4.15):

$$(6.2) \qquad q_2 = \frac{[illegible]}{2\pi} \iiint \frac{e^{[illegible]}}{r'} \, p \, dv - \frac{\lambda \, \gamma_0}{4\pi} \iint \frac{e^{[illegible]}}{r'} \, \frac{\partial [illegible]}{\partial n} \, [illegible] .$$

The quantity λ was defined in (4.12) and ρ_0 is the density of water. The first integral is taken over the volume filled with oil, the second over the surface separating oil and water.

We assume that $f(\varphi, \vartheta)$ in (6.1) is given by:

$$(6.3) \qquad f(\varphi, \vartheta) = \begin{cases} [illegible] & , \; 0 < \vartheta < \vartheta_0 \\ [illegible] & , \; \vartheta_0 < \vartheta \end{cases} .$$

i.e., that the original beam p is zero everywhere except in a solid sector of semi-angle ϑ_0, where it is given by $e^{ik_0 r}/r$. We assume also that the oil fills a sphere of radius $r = R$. In addition, we calculate the value of q_2 only on the axis of the original beam and at a large distance from the origin. This should furnish an estimate for the maximum disturbance that might be expected. The following figure indicates the situation, the cross-hatched volume being that from which the disturbances originate.

The quantities D and r' are distances to the point of integration, one being measured from the origin, the other from the point (x, y, z) at which q_2 is to be determined. Hence we may write

$$(6.4) \qquad p = \frac{e^{ik_0 D}}{D}, \quad \frac{\partial p}{\partial n} = \left(ik_0 - \frac{1}{D}\right)\frac{e^{ik_0 D}}{D},$$

and (6.2) can be written in our special case

(6.5) $$\psi_s = \frac{k_0}{2\pi} \iiint\limits_{\substack{[illegible] \leq r \leq R \\ [illegible] \leq \vartheta \leq \vartheta_0}} \frac{e^{ik_0 r'}}{r'} \frac{e^{ik_0 D}}{D} \, d[illegible]$$

$$[illegible] \; \frac{e^{ik_0 D}}{D} \iint\limits_{\substack{r = R \\ [illegible] \leq \vartheta \leq \vartheta_0}} \frac{e^{ik_0 r'}}{r'} \, [illegible]$$

We have

(6.6) $$r'^2 = r^2 + D^2 - 2rD \cos \vartheta \;,$$

from which we find, upon neglecting powers of D/r beyond the first:

(6.7) $$[illegible]$$

(6.8) $$\frac{1}{r'} = \frac{1}{r} \left(1 + \frac{D}{r} \cos \vartheta\right)$$

Once these approximations have been made it is found that the integrals in (6.5) can be evaluated explicitly to yield

(6.9) $$\psi_s = - \frac{e^{ik_0 r}}{r} \left\{ \frac{e^{ik_0 R(1 - \cos \vartheta_0)}}{k_0 (1 - \cos \vartheta_0)} - [illegible] \right.$$

$$\left. - [illegible] \left(1 + [illegible]\right) \left(e^{ik_0 R(1 - \cos \vartheta_0)} [illegible]\right) \right)$$

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If we set $\vartheta_0 = \pi$ we obtain the case of the full sphere: in this case the result of the exact solution given by equations (5.6) and (5.7) of the preceding section agrees exactly with (6.9).

From (6.9) we can obtain the relative amplitude [illegible]

$$\frac{\left| p + \kappa p_2 \right| - |p|}{|p|}$$

caused by the presence of oil. Just as in section 6 of AIP Memorandum No. 20.1, this quantity is given by Re $\left(\frac{\kappa p_2}{p}\right)$, to first order terms in κ, Re meaning that the real part of what follows is to be taken. For the relative amplitude disturbance in our case we find:

$$\text{(6.10)} \quad \text{Re}\left(\frac{\kappa p_2}{p}\right) = -\kappa \left\{ \left(\frac{\lambda \rho_0}{2} + \frac{1}{k_0(1-\cos\vartheta_0)} \right) \left(\cos\left[k_0 R(1-\cos\vartheta_0)\right] - \frac{\lambda \rho_0}{2 k_0 R} \sin\left[k_0 R\,(1-\cos\vartheta_0)\right] \right) \right\}$$

In the following table we give values for the relative amplitude disturbance Re $\left(\frac{\kappa p_2}{p}\right)$ for various oils (temperature 25°C) and for $\vartheta_0 = 15^\circ$ and 30°. The frequency assumed is 24,000 cycles/sec.

			$Re(\frac{Kq_2}{P})$		
Liquid Used	$k=\frac{\omega}{c}$	ρ	$\vartheta_0=15^\circ$	$\vartheta_0=30^\circ$	Full Spherical Wave $\vartheta_0=180^\circ$
Water	.985	1.03	0	0	0
Diethyl Phthalate	1.06	1.104	.467	.986	.016
Castor Oil	1.023	.969	.247	.599	.086
Liquid E.C.	0.937	1.073	-.309	-.635	-.088
Lubric Oil	0.985	.964	.016	.075	.069
Kerosene	1.14	.81	1.013	.567	.45

We observe that the relative amplitude disturbances, except for the case of Lubric Oil, are much larger for $\vartheta_0 = 15^\circ$ and $\vartheta_0 = 30^\circ$ than they are for the spherical wave. Even for Lubric Oil the relative amplitude disturbance would be rather large if the temperature were lowered down to about freezing. It is, however, somewhat doubtful in these cases whether our perturbation method furnishes accurate values when the relative disturbance is large, so that the values given in the above table may be subject to considerable errors. Nevertheless, the indications are that a sharp directional beam is more strongly affected by the use of oil inside a dome than is the case with a spherical wave, for example.

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